2008

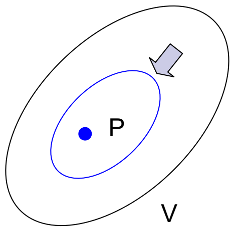
Advanced Disaster Control Engineering

Nagaoka University of Technology
Satoru Ohtsuka

(x,y,z,t)

Continuum body

- Continuous distribution of material in space and time
  (x,y,z,t) where (x,y,z)=space & t=time
- Density of body at point P
  $$\rho(P) = \lim_{V \to 0} \frac{M}{V}$$
  M=mass, V=volume
  existence of limit

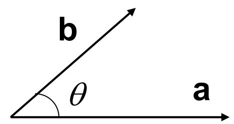


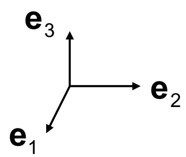
Newton's law of motion

- variation in momentum is equivalent to working force
  $$\frac{d}{dt}(m\dot{\mathbf{x}}) = \mathbf{f}$$
  where $\dot{\mathbf{x}}$ =velocity & $\mathbf{f}$ =force

Inner product of vectors

- Vector: direction and magnitude
- Inner product: projection, making scalar from vectors
  $$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$
- Length of vector
  $$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$$
- Normalized base vector: $\mathbf{e}_i$
  $$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$$





Length of vectors is unit & orthogonal

Description of vector

- Inner product

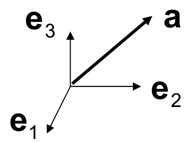
$$\mathbf{a}\cdot\mathbf{b} = (a_i\mathbf{e}_i)\cdot(b_j\mathbf{e}_j) = a_ib_j\mathbf{e}_i\cdot\mathbf{e}_j = a_ib_j\delta_{ij} = a_ib_i$$

- Component: projection of vector to base vector

$$a_i = \mathbf{a}\cdot\mathbf{e}_i = (a_j\mathbf{e}_j)\cdot\mathbf{e}_i = a_j\delta_{ji}$$

- Description of vector

$$\mathbf{a} = a_i\mathbf{e}_i = (\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}_i$$ i is a dummy index



Outer product of vectors

- Outer product

$$\mathbf{w} = \mathbf{a}\times\mathbf{b}$$

$\mathbf{w}$ is orthogonal to $\mathbf{a}$ & $\mathbf{b}$ and the magnitude equals to the are of parallelogram by $\mathbf{a}$ & $\mathbf{b}$.

$$\mathbf{w} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon_{ijk}\mathbf{e}_i a_j b_k$$

Permutation symbol $\varepsilon_{ijk} = \begin{cases} 1 & (\text{ijk : permutable}) \\ -1 & (\text{ijk : reverse}) \\ 0 & (\text{ijk : other}) \end{cases}$

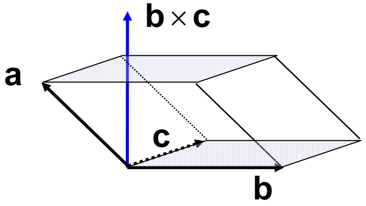
Scalar triple product of vectors

- Scalar triple product

$$[\mathbf{abc}] = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$$

indicates volume of parallelepiped by $\mathbf{a}$, $\mathbf{b}$ & $\mathbf{c}$.

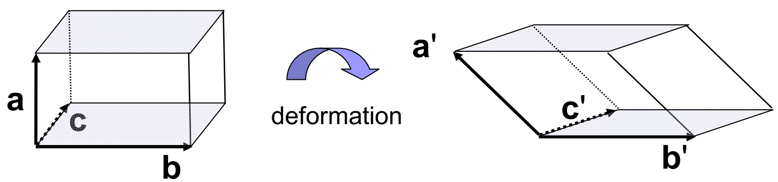
$$dV = [\mathbf{abc}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



Deformation

- Volume change

$\mathbf{a}' = \mathbf{F}\cdot\mathbf{a}$, $\mathbf{F}$:deformation gradient



$$dV' = [\mathbf{a}'\mathbf{b}'\mathbf{c}'] = [\mathbf{F}\cdot\mathbf{a}\ \mathbf{F}\cdot\mathbf{b}\ \mathbf{F}\cdot\mathbf{c}] = [\mathbf{F}][\mathbf{abc}] = (\det\mathbf{F})dV$$

Tensor

- Linear transformation from $\mathbf{a}$ to $\mathbf{b}$

$$\mathbf{b} = \mathbf{A} \cdot \mathbf{a}$$

- Transpose of tensor

$$\mathbf{a} \cdot (\mathbf{A} \cdot \mathbf{b}) = (\mathbf{A}^T \cdot \mathbf{a}) \cdot \mathbf{b}$$

- Symmetric and antisymmetric

$\mathbf{A} = \mathbf{A}^T$:symmetric, $\mathbf{A} = -\mathbf{A}^T$:antisymmetric

- Division of tensor

$$\mathbf{A} = \underbrace{\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(\mathbf{A} - \mathbf{A}^T)}_{\text{antisymmetric}}$$

Tensor product

- Definition

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

$\mathbf{c}$ is transformed to $(\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

$\mathbf{a} \otimes \mathbf{b}$ works as a tensor.

- Tensor description

$$\mathbf{A} = \mathbf{e}_i \otimes \mathbf{e}_j A_{ij}$$

Orthogonal tensor

- Transformation to keep inner product of vectors

$$(\mathbf{Q} \cdot \mathbf{a}) \cdot (\mathbf{Q} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$$

property $\mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{I}$ or $\mathbf{Q}^T = \mathbf{Q}^{-1}$

- Features:

keeping length

$$|\mathbf{Q} \cdot \mathbf{a}| = \sqrt{(\mathbf{Q} \cdot \mathbf{a}) \cdot (\mathbf{Q} \cdot \mathbf{a})} = \sqrt{(\mathbf{Q}^T \mathbf{Q} \cdot \mathbf{a}) \cdot \mathbf{a}} = \sqrt{\mathbf{a} \cdot \mathbf{a}} = |\mathbf{a}|$$

keeping angle between vectors

$$(\mathbf{Q} \cdot \mathbf{a}) \cdot (\mathbf{Q} \cdot \mathbf{b}) = |\mathbf{Q} \cdot \mathbf{a}||\mathbf{Q} \cdot \mathbf{b}|\cos\theta = |\mathbf{a}||\mathbf{b}|\cos\theta = \mathbf{a} \cdot \mathbf{b}$$

Eigenvalue and eigenvector

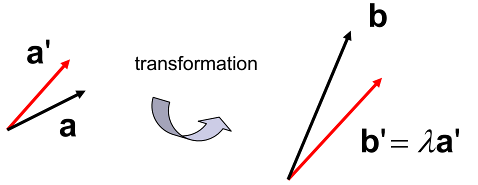
- Eigenvalue problem

$\mathbf{b} = \mathbf{A} \cdot \mathbf{a} = \lambda \mathbf{a}$ eigenvalue: λ , eigenvector: $\mathbf{a}$

- Eigen equation

$$(\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{a} = \mathbf{0}$$

condition to have a solution as $\mathbf{a} \neq \mathbf{0}$: $|\mathbf{A} - \lambda \mathbf{I}| = 0$



Eigenvalue and eigenvector

- Eigen equation

$$|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} A_{11}-\lambda & A_{12} & A_{13} \\ A_{21} & A_{22}-\lambda & A_{23} \\ A_{31} & A_{32} & A_{33}-\lambda \end{vmatrix} = 0$$

that is,

$$\boxed{\lambda^3 - I_A\lambda^2 + II_A\lambda - III_A = 0}$$ (eigen equation)

Eigen values are real in case of symmetric tensor, $\mathbf{A}$

Eigen equation

$$\boxed{\lambda^3 - I_A\lambda^2 + II_A\lambda - III_A = 0}$$ (eigen equation)

$$I_A = tr\mathbf{A} = A_{11} + A_{22} + A_{33} = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_A = \tfrac{1}{2}\left((tr\mathbf{A})^2 - tr\mathbf{A}^2\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$III_A = \det\mathbf{A} = \lambda_1\lambda_2\lambda_3$$

Invariants of tensor for coordinate transformation

$$\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \mathbf{Q} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$$

$\mathbf{Q}$:coordinate transformation by normalized eigen vectors

Cayley-Hamilton's law

- Applying a vector $\mathbf{a}$, eigen equation becomes

$$\left(\lambda^3 - I_A\lambda^2 + II_A\lambda - III_A\right)\mathbf{a} = \mathbf{0}$$

Using property of eigenvector $\mathbf{A}\cdot\mathbf{a} = \lambda\,\mathbf{a}$,

$$\left(\mathbf{A}^3 - I_A\mathbf{A}^2 + II_A\mathbf{A} - III_A\mathbf{I}\right)\mathbf{a} = \mathbf{0}$$

From $\mathbf{a} \neq \mathbf{0}$, $\boxed{\mathbf{A}^3 - I_A\mathbf{A}^2 + II_A\mathbf{A} - III_A\mathbf{I} = \mathbf{0}}$

- tensor function is expressed by

$$\boxed{f\left(\mathbf{I},\mathbf{A},\mathbf{A}^2,\cdots,\mathbf{A}^n\right) = f\left(\mathbf{I},\mathbf{A},\mathbf{A}^2\right)}$$

Constitutive equation

- Stress – strain relationship

power law

$$\sigma = f\left(\mathbf{I},\varepsilon,\varepsilon^2,\cdots,\varepsilon^n\right) = f\left(\mathbf{I},\varepsilon,\varepsilon^2\right)$$

Gradient

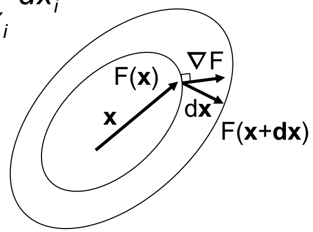
- Nabla: $\nabla = \mathbf{e}_i \dfrac{\partial}{\partial x_i} = \mathbf{e}_1 \dfrac{\partial}{\partial x_1} + \mathbf{e}_2 \dfrac{\partial}{\partial x_2} + \mathbf{e}_3 \dfrac{\partial}{\partial x_3}$

  example

$$dF = \frac{\partial F}{\partial x_i} dx_i \quad \text{where} \quad F = F(x_1, x_2, x_3)$$

$$\boxed{dF} = \nabla F \cdot d\mathbf{x} = \boxed{\left(\mathbf{e}_i \frac{\partial F}{\partial x_i}\right)} \cdot \boxed{\left(\mathbf{e}_j dx_j\right)} = \frac{\partial F}{\partial x_i} dx_i$$

variation gradient increment



Gradient & Deformation

- Gradient of vector

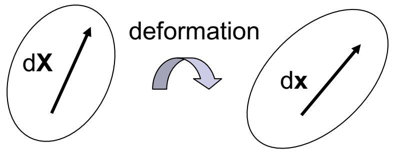
$$\nabla \mathbf{u} = \left(\mathbf{e}_i \frac{\partial}{\partial x_i}\right) \otimes \left(\mathbf{e}_j u_j\right) = \frac{\partial u_j}{\partial x_i} \mathbf{e}_i \otimes \mathbf{e}_j$$

- Deformation gradient

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = \left(F_{ij} \mathbf{e}_i \otimes \mathbf{e}_j\right) \cdot \left(dX_k \mathbf{e}_k\right) = F_{ik} dX_k \mathbf{e}_i = dx_i \mathbf{e}_i$$

$$F_{ik} = \frac{dx_i}{dX_k}$$

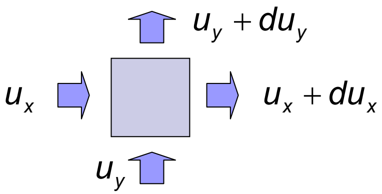
$$\mathbf{F} = \frac{dx_i}{dX_j} \mathbf{e}_i \otimes \mathbf{e}_j$$



Divergence

- Divergence of vector

$$\nabla \cdot \mathbf{u} = \operatorname{div} \mathbf{u} = \left(\mathbf{e}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\mathbf{e}_j u_j\right) = \frac{\partial u_j}{\partial x_i} \mathbf{e}_i \cdot \mathbf{e}_j = \frac{\partial u_i}{\partial x_i} = u_{i,i}$$



$$d\phi = du_x dy + du_y dx = \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) dA = u_{i,i} dA = div\, \mathbf{u}\, dA$$

Divergence

- Divergence of tensor

$$\nabla \cdot \mathbf{T} = \operatorname{div} \mathbf{T} = \left(\mathbf{e}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\mathbf{e}_j \otimes \mathbf{e}_k T_{jk}\right) = \mathbf{e}_k \frac{\partial T_{jk}}{\partial x_j} = \mathbf{e}_k T_{jk,j}$$

$$\mathbf{T} \cdot \nabla = \left(\mathbf{e}_j \otimes \mathbf{e}_k T_{jk}\right) \cdot \left(\mathbf{e}_i \frac{\partial}{\partial x_i}\right) = \mathbf{e}_j \frac{\partial T_{ji}}{\partial x_i} = \mathbf{e}_k \frac{\partial T_{kj}}{\partial x_j} = \mathbf{e}_k T_{kj,j}$$

Integral theorem

- Gauss's divergence theorem

$$\int_S \mathbf{n}\cdot\mathbf{F}\,dS = \int_V \nabla\cdot\mathbf{F}\,dV$$

$$\iint_S \left(F_x dydz + F_y dxdz + F_z dxdy\right) = \int_V \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dxdydz$$

- Generalized integral theorem

$$\int_S \mathbf{n} * \mathbf{F}\,dS = \int_V \nabla * \mathbf{F}\,dV$$

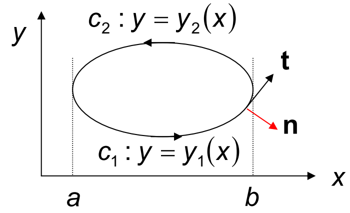
Proof (no.1)

- Gauss's divergence theorem (1 dimensional case)

$$\int_S n_i f\,dS = \int_V \frac{\partial f}{\partial x_i} dV$$

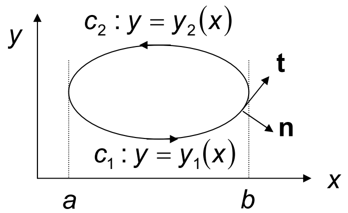
- proof

$$\int_S n_y f\,dS = \int_V \frac{\partial f}{\partial y} dV$$



$$\mathbf{t} = \begin{pmatrix} dx/ds \\ dy/ds \end{pmatrix} \qquad \mathbf{n} = \begin{pmatrix} dy/ds \\ -dx/ds \end{pmatrix}$$

Proof (no.2)



$$\mathbf{n} = \begin{pmatrix} dy/ds \\ -dx/ds \end{pmatrix}$$

$$\begin{aligned}
\int_S \frac{\partial f}{\partial y} dS &= \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \frac{\partial f}{\partial y} dy \right] dx \\
&= \int_a^b \{ yf(x, y_2(x)) - f(x, y_1(x)) \} dx \\
&= -\int_b^a f(x, y_2(x)) dx - \int_a^b f(x, y_1(x)) dx \\
&= -\int_c f\,dx = -\int_c f \frac{dx}{ds} ds = \int_c n_y f\,ds
\end{aligned}$$

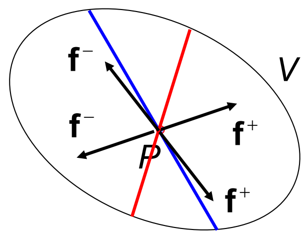
Stress

- Traction force

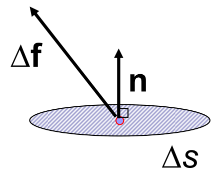
  By setting an arbitrary plane on point P, define a working traction force, $\mathbf{f}$.

- Action and reaction

$$\mathbf{f}^+ + \mathbf{f}^- = \mathbf{0}$$



- Stress vector



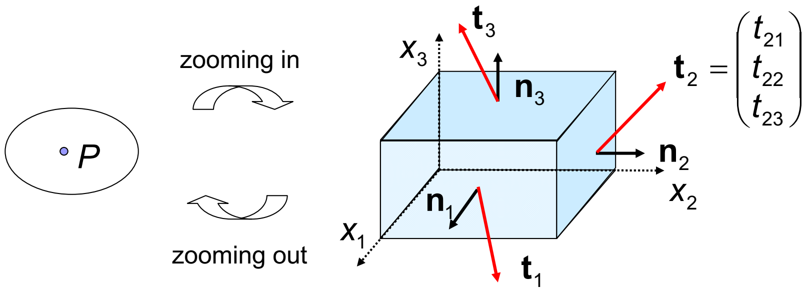
$$\mathbf{t} = \lim_{\Delta s \to 0} \frac{\Delta \mathbf{f}}{\Delta s}$$

$$\mathbf{t} = \mathbf{t}(\mathbf{x}, \mathbf{n})$$

Stress vector is a function of location $\mathbf{x}$ and intersecting plane with P (normal vector $\mathbf{n}$)

Stress tensor

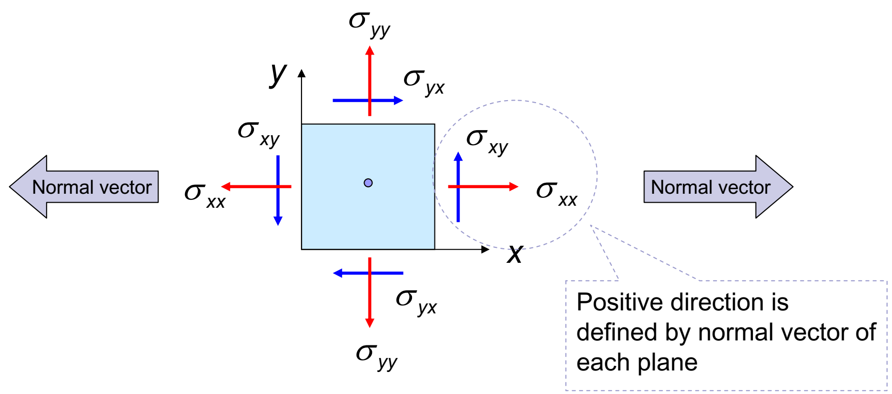
- Tensor composed of stress vectors



- Stress tensor

$$\sigma(\mathbf{x}) = \begin{pmatrix} \mathbf{t}_1^T \\ \mathbf{t}_2^T \\ \mathbf{t}_3^T \end{pmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

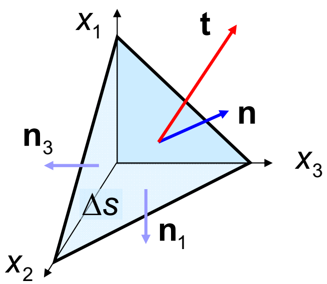
Note on positive direction of stress



Cauchy's theorem

- Area of coordinate planes

$$\Delta s_1 = \Delta s\,\mathbf{n}\cdot(-\mathbf{n}_1) \qquad \Delta v = \Delta s\, h/3$$
$$\Delta s_2 = \Delta s\,\mathbf{n}\cdot(-\mathbf{n}_2)$$
$$\Delta s_3 = \Delta s\,\mathbf{n}\cdot(-\mathbf{n}_3)$$



- Equilibrium equation of force

$$\rho\mathbf{b}\Delta v + \mathbf{t}\Delta s + \mathbf{t}_1\Delta s_1 + \mathbf{t}_2\Delta s_2 + \mathbf{t}_3\Delta s_3 = \mathbf{0}$$

$$\Rightarrow \rho\mathbf{b}\Delta s\frac{h}{3} + \mathbf{t}\Delta s + \mathbf{t}_1\Delta s\mathbf{n}\cdot(-\mathbf{n}_1) + \mathbf{t}_2\Delta s\mathbf{n}\cdot(-\mathbf{n}_2) + \mathbf{t}_3\Delta s\mathbf{n}\cdot(-\mathbf{n}_3) = \mathbf{0}$$

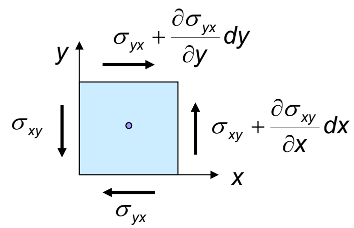
$$\Rightarrow \mathbf{t} = \mathbf{t}_1\mathbf{n}\cdot\mathbf{n}_1 + \mathbf{t}_2\mathbf{n}\cdot\mathbf{n}_2 + \mathbf{t}_3\mathbf{n}\cdot\mathbf{n}_3$$

$$\Rightarrow \mathbf{t} = \mathbf{n}\cdot(\mathbf{n}_1\otimes\mathbf{t}_1 + \mathbf{n}_2\otimes\mathbf{t}_2 + \mathbf{n}_3\otimes\mathbf{t}_3) = \mathbf{n}\cdot\begin{pmatrix} \mathbf{t}_1^T \\ \mathbf{t}_2^T \\ \mathbf{t}_3^T \end{pmatrix}$$

$$\Rightarrow \boxed{\mathbf{t} = \mathbf{n}\cdot\sigma}$$

Symmetry property of stress tensor

- Equilibrium equation of moment



$$\left(\sigma_{xy} + \left(\sigma_{xy} + \frac{\partial\sigma_{xy}}{\partial x}dx\right)\right)dy\frac{dx}{2} = \left(\sigma_{yx} + \left(\sigma_{yx} + \frac{\partial\sigma_{yx}}{\partial y}dy\right)\right)dx\frac{dy}{2}$$

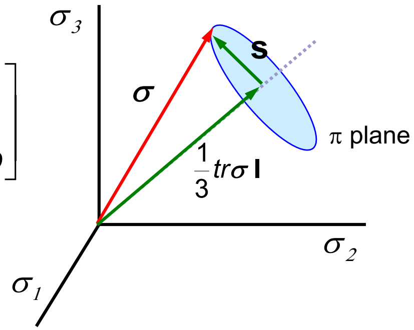
By $dx, dy \to 0$, $\boxed{\sigma_{xy} = \sigma_{yx}}$:symmetry

Deviator stress

- Definition

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3} tr\boldsymbol{\sigma}\,\mathbf{I} = \boldsymbol{\sigma} - p\,\mathbf{I}$$

$$\mathbf{s} = \begin{bmatrix} \sigma_{11} - p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - p \end{bmatrix}$$

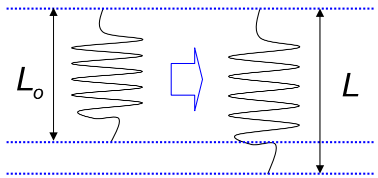


- Property

$$tr\mathbf{s} = 0$$

Strain

- Dimensionless quantity to express deformation



$$\varepsilon = \frac{L - L_o}{L_o} \qquad \varepsilon = \frac{L - L_o}{L}$$

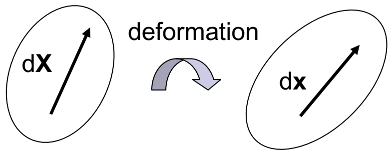
$$e = \frac{L^2 - L_o^2}{L_o^2} \qquad e = \frac{L^2 - L_o^2}{L^2}$$

various kinds of strain

Definition of strain

- Deformation gradient

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$$



- Expression of stretch

$$ds^2 - dS^2 = d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X} = (\mathbf{F} \cdot d\mathbf{X}) \cdot (\mathbf{F} \cdot d\mathbf{X}) - d\mathbf{X} \cdot d\mathbf{X}$$
$$= d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F} - \mathbf{I}) \cdot d\mathbf{X} = d\mathbf{X} \cdot 2\mathbf{E} \cdot d\mathbf{X}$$

$\mathbf{E}$: Green strain tensor $(\mathbf{E} = \mathbf{E}^T)$

$$ds^2 - dS^2 = d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{x} \cdot d\mathbf{x} - (\mathbf{F}^{-1} \cdot d\mathbf{x}) \cdot (\mathbf{F}^{-1} \cdot d\mathbf{x})$$
$$= d\mathbf{x} \cdot (\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}) \cdot d\mathbf{x} = d\mathbf{x} \cdot 2\mathbf{e} \cdot d\mathbf{x}$$

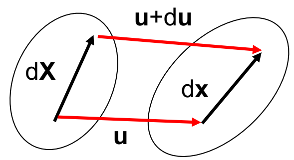
$\mathbf{e}$: Almansi strain tensor $(\mathbf{e} = \mathbf{e}^T)$

Green strain tensor

- Displacement vector

$$\mathbf{u} = \mathbf{x} - \mathbf{X}$$

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial(\mathbf{X} + \mathbf{u})}{\partial \mathbf{X}} = \mathbf{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$$



- Green strain tensor

$$\mathbf{E} = \frac{1}{2}\left\{ \frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^T + \boxed{\left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^T \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)} \right\}$$

second order → 0

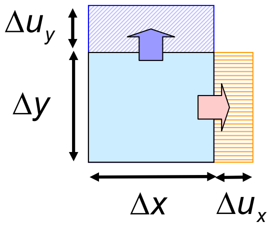
- Small strain tensor

$$\varepsilon = \frac{1}{2}\left\{ \frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^T \right\}$$

Geometrically based interpretation

- Stretching

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \qquad \varepsilon_y = \frac{\partial u_y}{\partial y}$$

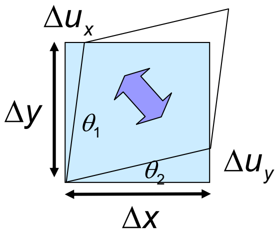


- Shearing

$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$= \tan\theta_1 + \tan\theta_2$$

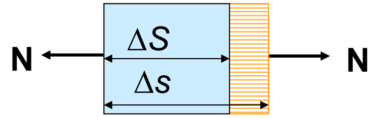
$$\approx \theta_1 + \theta_2$$



Discussion on strain No.1

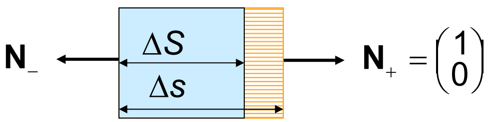
- Stretching

$$\frac{ds - dS}{dS} = \frac{ds}{dS} - 1 = \left(\frac{ds^2}{dS^2}\right)^{1/2} - 1$$



$$\left(\frac{ds}{dS}\right)^2 = \frac{d\mathbf{X}\cdot\mathbf{F}^T\mathbf{F}d\mathbf{X}}{d\mathbf{X}\cdot d\mathbf{X}} = \frac{\mathbf{N}\cdot\mathbf{F}^T\mathbf{F}\mathbf{N}}{\mathbf{N}\cdot\mathbf{N}} = \mathbf{N}\cdot\mathbf{F}^T\mathbf{F}\mathbf{N}$$

$$\mathbf{F}^T\mathbf{F} = \left(\mathbf{I} + \frac{\partial\mathbf{u}}{\partial\mathbf{X}}\right)^T\left(\mathbf{I} + \frac{\partial\mathbf{u}}{\partial\mathbf{X}}\right) = \mathbf{I} + 2\boldsymbol{\varepsilon}$$



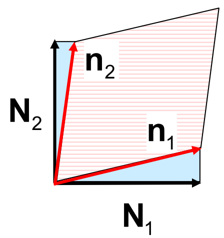
$$\frac{ds - dS}{dS} = \sqrt{\left\{\begin{pmatrix}1 & 0\end{pmatrix}\begin{bmatrix}1+2\varepsilon_{11} & 2\varepsilon_{12}\\ 2\varepsilon_{21} & 1+2\varepsilon_{22}\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\}} - 1$$

$$= \sqrt{1+2\varepsilon_{11}} - 1 \approx 1 + \frac{1}{2}\cdot 2\varepsilon_{11} - 1 = \boxed{\varepsilon_{11}}$$

Discussion on strain No.2

- Shearing

$$\cos(\mathbf{n}_1 \quad \mathbf{n}_2) = \frac{\mathbf{n}_1\cdot\mathbf{n}_2}{\sqrt{\mathbf{n}_1\cdot\mathbf{n}_1}\sqrt{\mathbf{n}_2\cdot\mathbf{n}_2}} = \frac{\mathbf{N}_1\cdot\mathbf{F}^T\mathbf{F}\mathbf{N}_2}{\sqrt{\mathbf{N}_1\cdot\mathbf{F}^T\mathbf{F}\mathbf{N}_1}\sqrt{\mathbf{N}_2\cdot\mathbf{F}^T\mathbf{F}\mathbf{N}_2}}$$



$$\cos(\mathbf{n}_1 \quad \mathbf{n}_2) = \frac{2\varepsilon_{12}}{\sqrt{1+2\varepsilon_{11}}\sqrt{1+2\varepsilon_{22}}}$$

$$\approx 2\varepsilon_{12}\left(1 - \frac{1}{2}\cdot 2\varepsilon_{11}\right)\left(1 - \frac{1}{2}\cdot 2\varepsilon_{22}\right)$$

$$= 2\varepsilon_{12} = \boxed{\gamma_{12}}$$

Compatibility condition on strain

- Strain & displacement

strain $\xrightarrow{\times}$ displacement (strain $\leftarrow$ displacement)

6 components — 3 components

- Compatibility condition is necessary to determine displacements from strains

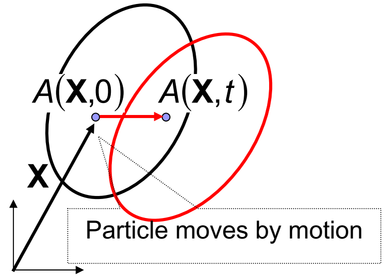
$$\varepsilon_x = \frac{\partial u_x}{\partial x} \qquad \varepsilon_y = \frac{\partial u_y}{\partial y} \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^2 u_x}{\partial x \partial^2 y} + \frac{\partial^2 u_y}{\partial^2 x \partial y}\right) = \frac{1}{2}\left(\frac{\partial^2 \varepsilon_x}{\partial^2 y} + \frac{\partial^2 \varepsilon_y}{\partial^2 x}\right)}$$

Material & spatial descriptions

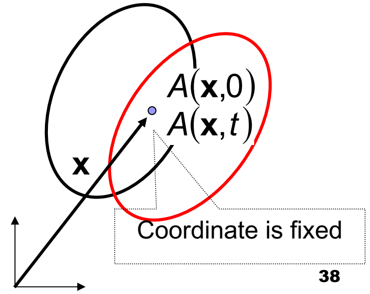
- Material description

looks at physical quantity of particle, $A(\mathbf{X},t)$ through the coordinate $\mathbf{X}$ at $t=0$ and time, t.



- Spatial description

looks at physical quantity of particle, $A(\mathbf{x},t)$ through the coordinate $\mathbf{x}$ at $t=t$ and time, t.



Conversion of description

- Position vector

description can be converted each other

$$\mathbf{x} = \mathbf{x}(\mathbf{X},t) \iff \mathbf{X} = \mathbf{X}(\mathbf{x},t)$$

- Change of description

$$A(\mathbf{X},t) = A(\mathbf{X}(\mathbf{x},t),t) \rightarrow A(\mathbf{x},t)$$

$$A(\mathbf{x},t) = A(\mathbf{x}(\mathbf{X},t),t) \rightarrow A(\mathbf{X},t)$$

Material time derivative

- Time rate of change in physical quantity of specific particle, $\mathbf{X}$

$$\dot{A} = \frac{\partial A(\mathbf{X},t)}{\partial t} = \lim_{\Delta t \to 0} \frac{A(\mathbf{X},t+\Delta t) - A(\mathbf{X},t)}{\Delta t}$$

- Velocity of particle

$$\dot{\mathbf{x}} = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}$$:Defined by material time derivative

It is expressed in spatial description such as

$$\mathbf{v}(\mathbf{x},t) = \dot{\mathbf{x}}(\mathbf{X}(\mathbf{x},t),t)$$

Spatial time derivative

- Time rate of change in physical quantity of particle at $\mathbf{x}$

$$\frac{\partial A(\mathbf{x},t)}{\partial t} = \lim_{\Delta t \to 0} \frac{A(\mathbf{x},t+\Delta t) - A(\mathbf{x},t)}{\Delta t}$$

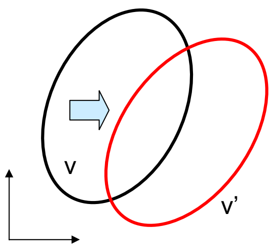
Relationship of two time derivatives

$$\begin{aligned}
\dot{A} &= \frac{\partial A(\mathbf{X},t)}{\partial t} \\
&= \lim_{\Delta t \to 0} \frac{A(\mathbf{x}(\mathbf{X},t+\Delta t),t+\Delta t) - A(\mathbf{X},t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{A(\mathbf{x}(\mathbf{X},t) + \mathbf{v}\Delta t, t+\Delta t) - A(\mathbf{X},t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{A(\mathbf{x}(\mathbf{X},t),t) + \nabla A \cdot \mathbf{v}\Delta t + \frac{\partial A}{\partial t}\Delta t - A(\mathbf{X},t)}{\Delta t}
\end{aligned}$$

$$\dot{A} = \frac{\partial A(\mathbf{X},t)}{\partial t} = \frac{\partial A}{\partial t} + \boxed{\nabla A \cdot \mathbf{v}}$$

Advection term

Time derivative of integral quantity



$$I(t) = \int_v A(\mathbf{x},t)\, dv$$

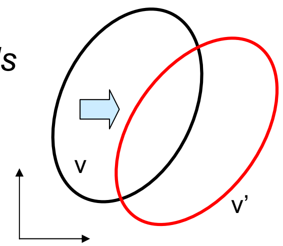
$$\begin{aligned}
\frac{DI}{Dt} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{v'} A(\mathbf{x},t+\Delta t)\, dv - \int_v A(\mathbf{x},t)\, dv\right] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_v (A(\mathbf{x},t+\Delta t) - A(\mathbf{x},t))\, dv + \int_{\Delta v} A(\mathbf{x},t+\Delta t)\, dv\right]
\end{aligned}$$

Preliminary for derivation

- 1 $\displaystyle \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_v (A(\mathbf{x},t+\Delta t) - A(\mathbf{x},t))\, dv\right] = \frac{\partial A}{\partial t}$

- 2 $\Delta v = (\mathbf{v}\cdot\mathbf{n})\, ds\Delta t$

$$\begin{aligned}
&\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\Delta v} A(\mathbf{x},t+\Delta t)\, dv \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_S A(\mathbf{x},t+\Delta t)(\mathbf{v}\cdot\mathbf{n})\Delta t\, ds \\
&= \int_S A\, (\mathbf{v}\cdot\mathbf{n})\, ds
\end{aligned}$$



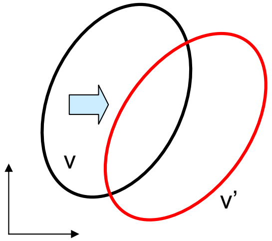
Time derivative of integral quantity

$$\frac{DI}{Dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_v \left(A(\mathbf{x},t+\Delta t) - A(\mathbf{x},t)\right) dv + \int_{\Delta v} A(\mathbf{x},t+\Delta t)\, dv\right]$$

$$\boxed{\frac{DI}{Dt} = \int_v \frac{\partial A}{\partial t} dv + \int_s A\,(\mathbf{v}\cdot\mathbf{n})ds}$$

or

$$\boxed{\frac{DI}{Dt} = \int_v \left\{\frac{\partial A}{\partial t} + \nabla\cdot(A\mathbf{v})\right\} dv}$$



Law of conservation of mass

- $I(t) = \int_v \rho(\mathbf{x},t)\, dv$
- $\boxed{\frac{DI}{Dt} = \int_v \frac{\partial \rho}{\partial t} dv + \int_s \rho\,(\mathbf{v}\cdot\mathbf{n})\, ds = 0}$

or

$$\boxed{\frac{DI}{Dt} = \int_v \left\{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right\} dv = 0}$$

Reynold's transport theorem

$$\frac{D}{Dt}\int_v A\rho\, dv = \left(\int_v A\rho\, dv\right)^{\bullet}$$

$$= \int_v \frac{\partial (A\rho)}{\partial t} dv + \int_s A\rho\,(\mathbf{v}\cdot\mathbf{n})\, ds$$

$$= \int_v \left[\frac{\partial (A\rho)}{\partial t} + \nabla\cdot(A\rho\mathbf{v})\right] dv$$

$$= \int_v \left[\rho\left\{\frac{\partial A}{\partial t} + \mathbf{v}\cdot\nabla A\right\} + A\left\{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right\}\right] dv$$

$$= \int_v \rho\dot{A}\;\; dv$$

Equilibrium equation

- Newton's law of motion

$$\frac{D}{Dt}\int_v \rho\mathbf{v}\, dv = \int_v \mathbf{f}\, dv + \int_s \mathbf{t}\, ds$$

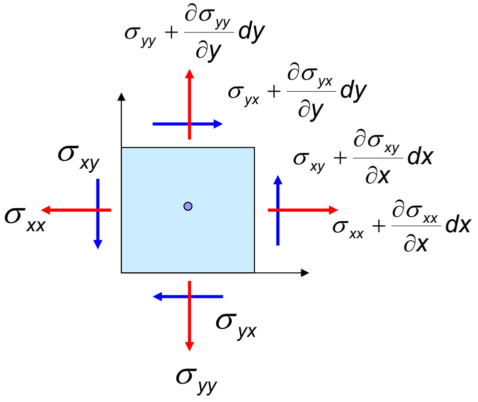
- Reynold's transport theorem & Cauchy's theorem

$$\frac{D}{Dt}\int_v \rho\mathbf{v}\, dv = \int_v \rho\dot{\mathbf{v}}\, dv\,, \quad \int_s \mathbf{t}\, ds = \int_s \mathbf{n}\cdot\boldsymbol{\sigma}\, ds = \int_v \nabla\cdot\boldsymbol{\sigma}\, dv$$

- Equilibrium equation

$$\boxed{\int_v \rho\dot{\mathbf{v}}\, dv = \int_v \{\nabla\cdot\boldsymbol{\sigma} + \mathbf{f}\}\, dv} \Rightarrow \boxed{\rho\dot{\mathbf{v}} = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f}}$$

Equilibrium equation



$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0$$

$$\Downarrow$$

$$\sigma_{ij,i} + f_j = 0$$

or

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}$$

Consideration on time derivative

- Volume of body

$$dV = \left| d\mathbf{X} \quad d\mathbf{Y} \quad d\mathbf{Z} \right|$$

- Deformation of body

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}, \quad d\mathbf{y} = \mathbf{F} \cdot d\mathbf{Y}, \quad d\mathbf{z} = \mathbf{F} \cdot d\mathbf{Z}$$

$$dv = \left| d\mathbf{x} \quad d\mathbf{y} \quad d\mathbf{z} \right| = \left| \mathbf{F} \cdot d\mathbf{X} \quad \mathbf{F} \cdot d\mathbf{Y} \quad \mathbf{F} \cdot d\mathbf{Z} \right| = |\mathbf{F}| dV$$

- Time derivative of dv

$$(dv)^{\bullet} = (|\mathbf{F}|)^{\bullet} dV + |\mathbf{F}| dV^{\bullet} = (|\mathbf{F}|)^{\bullet} dV = |\mathbf{F}| tr\left(\mathbf{F}^{-1}\dot{\mathbf{F}}\right) dV$$

$$\mathbf{F}^{-1}\dot{\mathbf{F}} = \frac{d\mathbf{X}}{d\mathbf{x}} \frac{d\dot{\mathbf{x}}}{d\mathbf{X}} = \frac{d\dot{\mathbf{x}}}{d\mathbf{x}} = \nabla \mathbf{v}$$

$$(dv)^{\bullet} = |\mathbf{F}| tr\left(\mathbf{F}^{-1}\dot{\mathbf{F}}\right) dV = |\mathbf{F}| (\nabla \cdot \mathbf{v}) dV = (\nabla \cdot \mathbf{v}) dv$$

Time derivative of integral

- Law of conservation of mass

$$\frac{D}{Dt} \int \rho \, dv = \int (\rho \, dv)^{\bullet} = \int \dot{\rho} \, dv + \int \rho \nabla \cdot \mathbf{v} \, dv = \boxed{\int \{\dot{\rho} + \rho \nabla \cdot \mathbf{v}\} dv = 0}$$

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v} + \rho \nabla \cdot \mathbf{v} = \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$

- Transport theorem

$$\frac{D}{Dt} \int \rho A \, dv = \int (\rho A \, dv)^{\bullet} = \int \dot{\rho} A + \rho \dot{A} \, dv + \int \rho A \nabla \cdot \mathbf{v} \, dv$$

$$= \int \{\rho \dot{A} + A(\dot{\rho} + \rho \nabla \cdot \mathbf{v})\} dv = \int \rho \dot{A} \, dv$$

Framework of boundary value problem

- Equilibrium equation

$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}$ in V , $\quad \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{t}$ on S

- Strain displacement relationship

$\boldsymbol{\varepsilon} = \frac{1}{2}\left\{\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right\}$ in V

- Constitutive equation

$\boldsymbol{\sigma} = C : \boldsymbol{\varepsilon}$ in V

- Boundary condition

$\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{t}_o$ on Sσ, $\quad \mathbf{u} = \mathbf{u}_o$ on Su

Principle of virtual work <case 1>

- If the following equation,

$$\int_V \sigma : \beta \, dv = \int_V \mathbf{f} \cdot \mathbf{w} \, dv + \int_{S\sigma} \mathbf{t}_o \cdot \mathbf{w} \, ds$$

is established for arbitrary variable $\mathbf{w}$ which satisfies

$$\begin{cases} \beta = \dfrac{1}{2}\left\{\nabla \mathbf{w} + (\nabla \mathbf{w})^T\right\} & \text{in V} \\ \mathbf{w} = \mathbf{0} & \text{on Su} \end{cases}$$

the equation reduces to equilibrium equation such that

$$\begin{cases} \nabla \cdot \sigma + \mathbf{f} = \mathbf{0} & \text{in V} \\ \mathbf{n} \cdot \sigma = \mathbf{t}_o & \text{on S}\sigma \end{cases}$$

From the condition

$$\int_V \sigma : \beta \, dv = \int_V \sigma : \frac{1}{2}\left(\nabla \mathbf{w} + \nabla \mathbf{w}^T\right) dv = \int_V \sigma : \nabla \mathbf{w} \, dv$$

$$= \int_V \nabla \cdot (\sigma \cdot \mathbf{w}) \, dv - \int_V \nabla \cdot \sigma \cdot \mathbf{w} \, dv$$

$$= \int_S \mathbf{n} \cdot \sigma \cdot \mathbf{w} \, ds - \int_V \nabla \cdot \sigma \cdot \mathbf{w} \, dv$$

$$= \int_S \mathbf{t} \cdot \mathbf{w} \, ds - \int_V \nabla \cdot \sigma \cdot \mathbf{w} \, dv$$

The equation becomes

$$\int_V \sigma : \beta \, dv - \int_V \mathbf{f} \cdot \mathbf{w} \, dv - \int_{S\sigma} \mathbf{t}_0 \cdot \mathbf{w} \, ds$$

$$= \int_S \mathbf{t} \cdot \mathbf{w} \, ds - \int_V \nabla \cdot \sigma \cdot \mathbf{w} \, dv - \int_V \mathbf{f} \cdot \mathbf{w} \, dv - \int_{S\sigma} \mathbf{t}_0 \cdot \mathbf{w} \, ds$$

$$= -\int_V (\nabla \cdot \sigma + \mathbf{f}) \cdot \mathbf{w} \, dv + \int_{S\sigma} (\mathbf{t} - \mathbf{t}_0) \cdot \mathbf{w} \, ds = 0 \quad \text{for any } \mathbf{w}$$

It reduces to

$$\Rightarrow \begin{cases} \nabla \cdot \sigma + \mathbf{f} = \mathbf{0} & \text{in V} \\ \mathbf{n} \cdot \sigma = \mathbf{t}_o & \text{on S}\sigma \end{cases}$$

Principle of virtual work <case 2>

- If the following equation,

$$\int_V \alpha : \varepsilon \, dv = \int_V \mathbf{f} \cdot \mathbf{u} \, dv + \int_{Su} \mathbf{t} \cdot \mathbf{u}_o \, ds$$

is established for arbitrary variable α which satisfies

$$\begin{cases} \nabla \cdot \alpha + \mathbf{f} = \mathbf{0} & \text{in V} \\ \mathbf{n} \cdot \alpha = \mathbf{0} & \text{on S}\sigma \end{cases}$$

the equation reduces to equilibrium equation such that

$$\begin{cases} \varepsilon = \dfrac{1}{2}\left\{\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right\} & \text{in V} \\ \mathbf{u} = \mathbf{u}_o & \text{on S}\sigma \end{cases}$$

From the condition

$$\int_V \mathbf{f} \cdot \mathbf{u} \, dv + \int_{Su} \mathbf{t} \cdot \mathbf{u}_o \, ds$$

$$= -\int_V \nabla \cdot \alpha \cdot \mathbf{u} \, dv + \int_{Su} \mathbf{t} \cdot \mathbf{u}_o \, ds$$

$$= -\int_V \nabla \cdot (\alpha \cdot \mathbf{u}) \, dv + \int_V \alpha : \nabla \mathbf{u} \, dv + \int_{Su} \mathbf{t} \cdot \mathbf{u}_o \, ds$$

$$= -\int_V \nabla \cdot (\alpha \cdot \mathbf{u}) \, dv + \int_V \alpha : \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) dv + \int_{Su} \mathbf{t} \cdot \mathbf{u}_o \, ds$$

The equation becomes

$$\int_V \alpha : \varepsilon \, dv - \int_V \mathbf{f} \cdot \mathbf{u} \, dv - \int_{Su} \mathbf{t} \cdot \mathbf{u}_o \, ds$$

$$= \int_V \alpha : \varepsilon \, dv + \int_V \nabla \cdot (\alpha \cdot \mathbf{u}) \, dv - \int_V \alpha : \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) dv - \int_{Su} \mathbf{t} \cdot \mathbf{u}_0 \, ds$$

$$= \int_V \alpha : \boxed{\left\{\varepsilon - \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)\right\}} dv + \int_{Su} \mathbf{n} \cdot \alpha \cdot \boxed{(\mathbf{u} - \mathbf{u}_0)} ds = 0$$

for any α

Elasticity & Plasticity

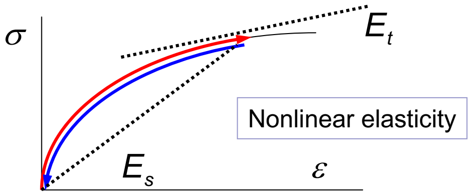
- Recoverable response: Elasticity

$\sigma = E_s \, \varepsilon$

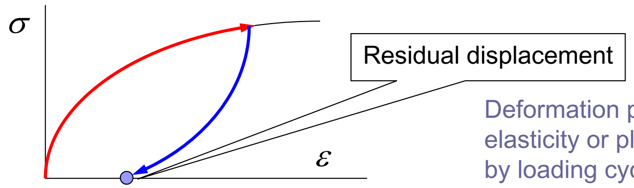
E_s :secant elastic modulus

$\Delta\sigma = E_t \, \Delta\varepsilon$

E_t :tangential elastic modulus



- Irrecoverable response: Plasticity



Deformation property whether elasticity or plasticity is judged by loading cycle.

Yield function and loading condition

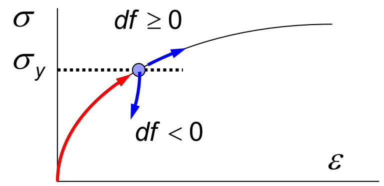
- Limit load for elasticity

Yield function: $f(\sigma) = \sigma - \sigma_y$

$f(\sigma) < 0$: elasticity $\quad \dot{\varepsilon}^p = 0$

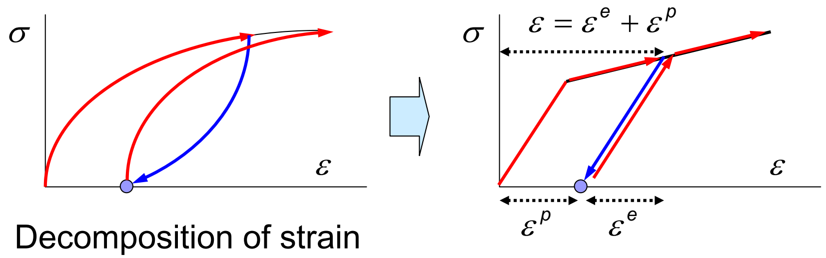
$f(\sigma) = 0$

$$\begin{cases} df = \dfrac{\partial f}{\partial \sigma} d\sigma \geq 0 & \text{: plasticity, strain hardening (}\sigma_y\text{ increases)} \quad \dot{\varepsilon}^p \neq 0 \\ df = \dfrac{\partial f}{\partial \sigma} d\sigma < 0 & \text{: elasticity} \quad \dot{\varepsilon}^p = 0 \end{cases}$$



Basic assumption on strain

- Modeling on elastic unloading behavior



- Decomposition of strain

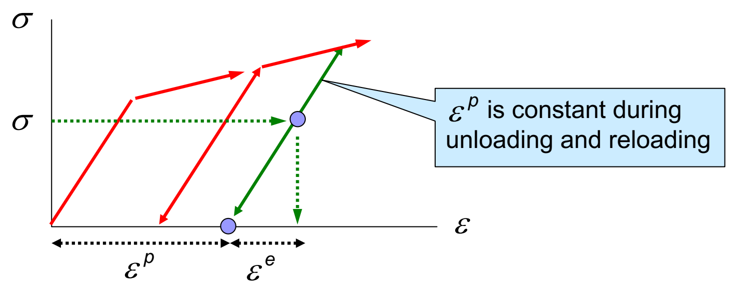
$\varepsilon = \varepsilon^e + \varepsilon^p$

to establish 1 to 1 relationship between stress and strain.

If ε^p is given, ε^e is unique for σ.

Consideration on assumption

- Uniqueness in stress & strain

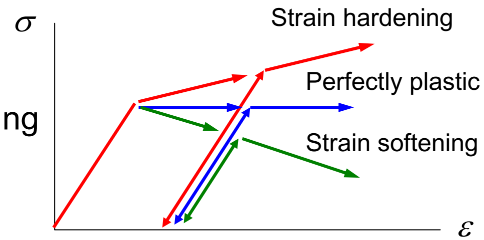


- Plasticity

makes boundary value problem in framework of elasticity.

Plastic model

- Perfectly plastic
  $f(\sigma)=0$
- Strain hardening & softening
  $f(\sigma,\kappa)=0$
- Hardening parameter
  $\kappa=\kappa(\varepsilon^p)$

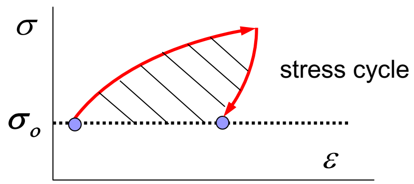


Basic assumption of plasticity

- Drucker's postulate:

  Plastic work is defined to be positive.

  $\oint(\sigma-\sigma_o)\cdot d\varepsilon \geq 0,$ σ_o :initial stress (arbitrary)

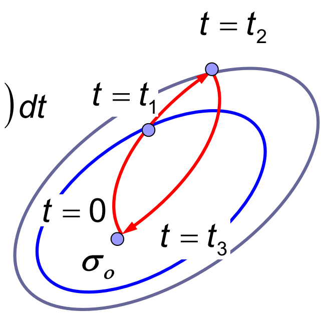


It is necessary to make 1 to 1 relationship between stress and strain.

Principle of maximum plastic work

- Drucker's postulate

$$\oint(\sigma-\sigma_o)\cdot d\varepsilon$$
$$=\int_0^{t_1}(\sigma-\sigma_o)\cdot\dot{\varepsilon}^e\,dt+\int_{t_1}^{t_2}(\sigma-\sigma_o)\cdot(\dot{\varepsilon}^e+\dot{\varepsilon}^p)dt$$
$$+\int_{t_2}^{t_3}(\sigma-\sigma_o)\cdot\dot{\varepsilon}^e\,dt$$
$$=\cancel{\int_0^{t_3}(\sigma-\sigma_o)\cdot\dot{\varepsilon}^e\,dt}+\int_{t_1}^{t_2}(\sigma-\sigma_o)\cdot\dot{\varepsilon}^p\,dt$$
$$=\int_{t_1}^{t_2}(\sigma-\sigma_o)\cdot\dot{\varepsilon}^p\,dt\geq 0$$

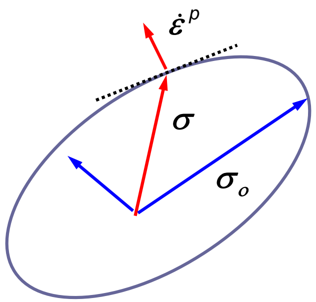


$\Rightarrow$ $\boxed{(\sigma-\sigma_o)\cdot\dot{\varepsilon}^p\geq 0}$ Principle of maximum plastic work

Principle of maximum plastic work

- Physical meaning

  $(\sigma-\sigma_o)\cdot\dot{\varepsilon}^p\geq 0$ $\Rightarrow$ $\sigma\cdot\dot{\varepsilon}^p\geq\sigma_o\cdot\dot{\varepsilon}^p$



σ_o :arbitrary possible stress

Associated flow rule

- Principle of maximum plastic work

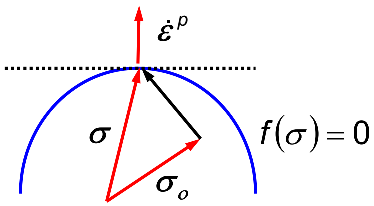
$$\boxed{(\sigma - \sigma_o)\cdot \dot{\varepsilon}^p \geq 0}$$

$$\Downarrow$$

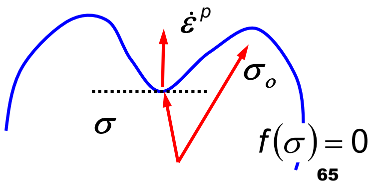
- Normality rule

$$\dot{\varepsilon}^p = \lambda \frac{\partial f}{\partial \sigma} \qquad \lambda \text{ :plastic multiplier}$$

λ is positive, but substantially indeterminate.



- Convexity of yield function



Equation for perfectly plastic body

- Associated flow rule $\quad d\varepsilon^p = d\lambda \dfrac{\partial f}{\partial \boldsymbol{\sigma}}$
- Perfectly plastic $\quad f(\sigma) = 0$, $\quad df = \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T d\boldsymbol{\sigma} = 0$
- Elastic $\quad d\boldsymbol{\sigma} = D d\varepsilon^e = D(d\varepsilon - d\varepsilon^p)$
- Arrangement $\quad \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T d\boldsymbol{\sigma} = \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T D d\varepsilon - \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T D \dfrac{\partial f}{\partial \boldsymbol{\sigma}} d\lambda = 0$
- Plastic multiplier

$$d\lambda = \frac{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T D d\varepsilon}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T D \dfrac{\partial f}{\partial \boldsymbol{\sigma}}}$$

Note: Conversion of tensor into vector

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} \dfrac{\partial f}{\partial \sigma_{xx}} & \dfrac{\partial f}{\partial \sigma_{xy}} & \dfrac{\partial f}{\partial \sigma_{xz}} \\ \dfrac{\partial f}{\partial \sigma_{yx}} & \dfrac{\partial f}{\partial \sigma_{yy}} & \dfrac{\partial f}{\partial \sigma_{yz}} \\ \dfrac{\partial f}{\partial \sigma_{zx}} & \dfrac{\partial f}{\partial \sigma_{zy}} & \dfrac{\partial f}{\partial \sigma_{zz}} \end{bmatrix} \Rightarrow \frac{\partial f}{\partial \boldsymbol{\sigma}} = \begin{pmatrix} \dfrac{\partial f}{\partial \sigma_{xx}} \\ \dfrac{\partial f}{\partial \sigma_{yy}} \\ \dfrac{\partial f}{\partial \sigma_{zz}} \\ \dfrac{\partial f}{\partial \sigma_{xy}} \\ \dfrac{\partial f}{\partial \sigma_{yz}} \\ \dfrac{\partial f}{\partial \sigma_{zx}} \end{pmatrix}$$

Perfectly plastic body

- Constitutive equation

$$d\boldsymbol{\sigma} = D d\varepsilon - D d\varepsilon^p = D d\varepsilon - D \frac{\partial f}{\partial \boldsymbol{\sigma}} d\lambda$$

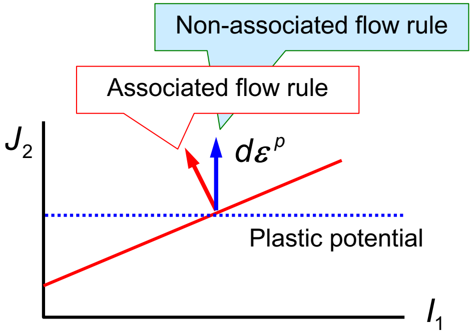
$$d\boldsymbol{\sigma} = D^{ep} d\varepsilon = \left[D - \frac{D \dfrac{\partial f}{\partial \boldsymbol{\sigma}} \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T D}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T D \dfrac{\partial f}{\partial \boldsymbol{\sigma}}} \right] d\varepsilon$$

Non-associated flow rule

- To express the material property, non-associated flow rule is required

$$d\varepsilon^p = d\lambda \frac{\partial g}{\partial \sigma}$$

- Yield function $f(\sigma)=0$
- Plastic potential $g(\sigma)=0$



Equation for perfectly plastic body

- Non-associated flow rule $d\varepsilon^p = d\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}}$
- Yield function $f(\sigma)=0$, $df = \left(\frac{\partial f}{\partial \sigma}\right)^T d\sigma = 0$
- Elastic $d\sigma = Dd\varepsilon^e = D(d\varepsilon - d\varepsilon^p)$
- Arrangement $\left(\frac{\partial f}{\partial \sigma}\right)^T d\sigma = \left(\frac{\partial f}{\partial \sigma}\right)^T Dd\varepsilon - \left(\frac{\partial f}{\partial \sigma}\right)^T D\frac{\partial g}{\partial \sigma} d\lambda = 0$
- Plastic multiplier $d\lambda = \dfrac{\left(\frac{\partial f}{\partial \sigma}\right)^T Dd\varepsilon}{\left(\frac{\partial f}{\partial \sigma}\right)^T D\frac{\partial g}{\partial \sigma}}$

Perfectly plastic body

- Constitutive equation of non-associated flow rule

$$d\sigma = Dd\varepsilon - Dd\varepsilon^p = Dd\varepsilon - D\frac{\partial g}{\partial \sigma} d\lambda$$

$$d\sigma = D^{ep} d\varepsilon = \left[D - \frac{D\frac{\partial g}{\partial \sigma}\left(\frac{\partial f}{\partial \sigma}\right)^T D}{\left(\frac{\partial f}{\partial \sigma}\right)^T D\frac{\partial g}{\partial \sigma}} \right] d\varepsilon$$

Equation for strain hardening body

- Associated flow rule $d\varepsilon^p = d\lambda \frac{\partial f}{\partial \sigma}$
- Yield function $f(\sigma, \varepsilon^p) = 0$
- Loading condition

$$\left(\frac{\partial f}{\partial \sigma}\right)^T d\sigma + \left(\frac{\partial f}{\partial \varepsilon^p}\right)^T d\varepsilon^p = \left(\frac{\partial f}{\partial \sigma}\right)^T d\sigma + \left(\frac{\partial f}{\partial \varepsilon^p}\right)^T \frac{\partial f}{\partial \sigma} d\lambda = 0$$

- Elastic $d\sigma = Dd\varepsilon^e = D(d\varepsilon - d\varepsilon^p)$
- Arrangement

$$\left(\frac{\partial f}{\partial \sigma}\right)^T d\sigma = \left(\frac{\partial f}{\partial \sigma}\right)^T Dd\varepsilon - \left(\frac{\partial f}{\partial \sigma}\right)^T D\frac{\partial f}{\partial \sigma} d\lambda = -\left(\frac{\partial f}{\partial \varepsilon^p}\right)^T \frac{\partial f}{\partial \sigma} d\lambda$$

Strain hardening body

- Plastic multiplier

$$d\lambda = \frac{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T D d\varepsilon}{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T D \frac{\partial f}{\partial \boldsymbol{\sigma}} - \left(\frac{\partial f}{\partial \boldsymbol{\varepsilon}^p}\right)^T \frac{\partial f}{\partial \boldsymbol{\sigma}}}$$

- Constitutive equation

$$d\boldsymbol{\sigma} = D^{ep} d\varepsilon = \left[D - \frac{D \frac{\partial f}{\partial \boldsymbol{\sigma}} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T D}{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T D \frac{\partial f}{\partial \boldsymbol{\sigma}} - \left(\frac{\partial f}{\partial \boldsymbol{\varepsilon}^p}\right)^T \frac{\partial f}{\partial \boldsymbol{\sigma}}} \right] d\varepsilon$$

Yield function

- von Mises criteria

$$f(\sigma) = J_2 - \sigma_o^2 = 0$$

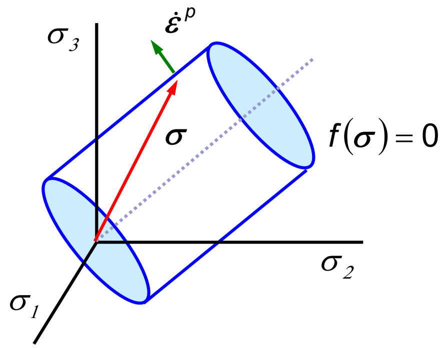
2nd invariant of deviator stress

$$J_2 = \frac{1}{2} \mathbf{s} : \mathbf{s} = \frac{1}{2} s_{ij} s_{ij}$$

- Associated flow rule

$$\dot{\boldsymbol{\varepsilon}}^p = \lambda \frac{\partial f}{\partial \boldsymbol{\sigma}} = \lambda \mathbf{s}$$

$$\Rightarrow \quad \dot{\boldsymbol{\varepsilon}}^p = \frac{\dot{e}}{\sqrt{2}\sigma_o} \mathbf{s} \quad \text{where} \quad \dot{e} = \sqrt{\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p} = \sqrt{\dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p}$$



Rigid plastic constitutive equation

- Associated flow rule for von Mises criteria

$$\mathbf{s} = \frac{\sqrt{2}\sigma_o}{\dot{e}} \dot{\boldsymbol{\varepsilon}}^p \quad \text{where} \quad \dot{e} = \sqrt{\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p} = \sqrt{\dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p}$$

- Constitutive equation (stress–strain rate relationship)

$$\boldsymbol{\sigma} = \mathbf{s} + p\mathbf{I} = \frac{\sqrt{2}\sigma_o}{\dot{e}} \dot{\boldsymbol{\varepsilon}}^p + p\mathbf{I}$$

p is nondeterministic for plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$.

It is determined by solving the boundary value problem (i.e. equilibrium equation). It is, however, not so easy. It will be noted later.

Yield function

- Extended von Mises criteria (Drucker-Prager)

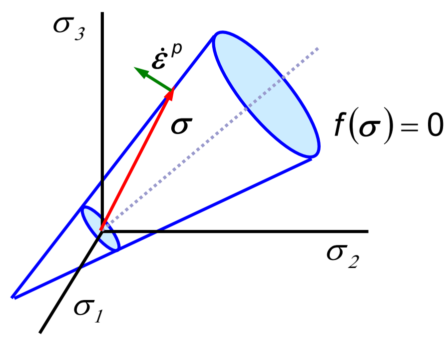
$$f(\sigma) = J_2 - \alpha I_1 - k = 0$$

1st invariant of stress

$$I_1 = tr\boldsymbol{\sigma}$$

- Associated flow rule

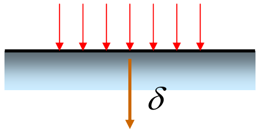
$$\dot{\boldsymbol{\varepsilon}}^p = \lambda \frac{\partial f}{\partial \boldsymbol{\sigma}} = \lambda(\mathbf{s} + \alpha \mathbf{I})$$

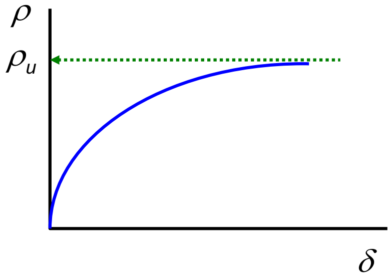


Ultimate load analysis (Plasticity theorem)

- Setting of problem

$\mathbf{F} = \rho \mathbf{F}_o$ $\quad$ $\mathbf{F}_o$:basic load, ρ :load factor





- Analytical methods
  - □ Deformation analysis: stiffness → 0 (rigorous analysis)
  - □ Limit analysis: $\Delta \mathbf{F} = \mathbf{0}$ but $\Delta \mathbf{u} \neq \mathbf{0}$

Limit analysis

- Upper & Lower bound theorems

  Two systems based on following conditions:
  - □ Equilibrium equation □ — Lower bound
  - □ Strain rate and velocity relationship □ — Upper bound
  - □ Constitutive equation □ □
  - □ Boundary condition □ □
- Property of solutions

$$\cdots \rho_L \le (\rho_L)_{\max} \le \rho_{exact} \le (\rho_U)_{\min} \le \rho_U \cdots$$

various solutions $\qquad$ various solutions

Plasticity theorem 1

- Stress distribution keeps constant at failure
- Proof

  Consider a body collapsing at load $\rho \mathbf{T}_o$

  $$\int_v \sigma : \dot{\varepsilon}\, dv = \int_v \mathbf{f} \cdot \dot{\mathbf{u}}\, dv + \int_{S\sigma} \rho \mathbf{t}_o \cdot \dot{\mathbf{u}}\, ds \qquad \text{(Work equation)}$$

  Assume stress changes under constant load $\rho \mathbf{T}_o$

  $$\int_v (\sigma + \dot{\sigma}\Delta t) : \dot{\varepsilon}\, dv = \int_v \mathbf{f} \cdot \dot{\mathbf{u}}\, dv + \int_{S\sigma} \rho \mathbf{t}_o \cdot \dot{\mathbf{u}}\, ds \qquad \text{(Virtual work equation)}$$

  From the above equations

  $\int_v \dot{\sigma} : \dot{\varepsilon}\, dv \Delta t = 0$ is necessary, that is $\dot{\sigma} : \dot{\varepsilon} = 0$

Continual proof

- Examination on $\dot{\sigma} : \dot{\varepsilon} = 0$

  $$\dot{\sigma} : \dot{\varepsilon} = \dot{\sigma} : \dot{\varepsilon}^e + \dot{\sigma} : \dot{\varepsilon}^p = 0$$

  where

  $\dot{\sigma} : \dot{\varepsilon}^e = \dot{\varepsilon}^e : \mathrm{A} : \dot{\varepsilon}^e \ge 0$:elastic energy

  $\dot{\sigma} : \dot{\varepsilon}^p \ge 0$:Drucker's postulate

  From the condition of $\dot{\sigma} : \dot{\varepsilon}^e = 0$ & $\dot{\sigma} : \dot{\varepsilon}^p = 0$,

  it is obtained as $\dot{\sigma} = \mathbf{0}$.

Plasticity theorem 2

- Stress & strain rate distribution is unique for external load
- Proof

Consider two sets of solutions as $\left(\sigma_{(1)},\dot{\varepsilon}^p_{(1)}\right)$ & $\left(\sigma_{(2)},\dot{\varepsilon}^p_{(2)}\right)$

$\sigma=\sigma_{(1)}-\sigma_{(2)}$:self-equilibrate

$\dot{\varepsilon}=\dot{\varepsilon}^p_{(1)}-\dot{\varepsilon}^p_{(2)}$:compatible with $\mathbf{u}=\mathbf{0}$ on S_σ

It is obtained that $\int_V\left(\sigma_{(1)}-\sigma_{(2)}\right):\left(\dot{\varepsilon}^p_{(1)}-\dot{\varepsilon}^p_{(2)}\right)dv=0$

Integrand is

$\left(\sigma_{(1)}-\sigma_{(2)}\right):\left(\dot{\varepsilon}^p_{(1)}-\dot{\varepsilon}^p_{(2)}\right)=\left(\sigma_{(1)}-\sigma_{(2)}\right):\dot{\varepsilon}^p_{(1)}+\left(\sigma_{(2)}-\sigma_{(1)}\right):\dot{\varepsilon}^p_{(2)}\geq 0$,

therefore, $\sigma_{(1)}=\sigma_{(2)}$.

Plasticity theorem 3

- Stress distribution approaches to specific one for load
- Proof

Consider two initial stress distributions as $\sigma_{(1)}(t=0)$ & $\sigma_{(2)}(t=0)$

$\sigma=\sigma_{(1)}-\sigma_{(2)}$:self-equilibrate

$\dot{\varepsilon}=\dot{\varepsilon}_{(1)}-\dot{\varepsilon}_{(2)}$:compatible with $\mathbf{u}=\mathbf{0}$ on S_σ

It is obtained that

$$\int_V\left(\sigma_{(1)}-\sigma_{(2)}\right):\left(\dot{\varepsilon}_{(1)}-\dot{\varepsilon}_{(2)}\right)dv=0$$

Continual proof

- Elastic energy

$$A=\frac{1}{2}\left(\sigma_{(1)}-\sigma_{(2)}\right):\mathrm{D}:\left(\sigma_{(1)}-\sigma_{(2)}\right)\geq 0$$

$$\dot{A}=\left(\sigma_{(1)}-\sigma_{(2)}\right):\mathrm{D}:\left(\dot{\sigma}_{(1)}-\dot{\sigma}_{(2)}\right)=\left(\sigma_{(1)}-\sigma_{(2)}\right):\left(\dot{\varepsilon}^e_{(1)}-\dot{\varepsilon}^e_{(2)}\right)$$

- State change

$$\int_V\left(\sigma_{(1)}-\sigma_{(2)}\right):\left(\dot{\varepsilon}_{(1)}-\dot{\varepsilon}_{(2)}\right)dv=0$$

$$=\int_V\dot{A}\,dv+\boxed{\int_V\left(\sigma_{(1)}-\sigma_{(2)}\right):\left(\dot{\varepsilon}^p_{(1)}-\dot{\varepsilon}^p_{(2)}\right)dv}$$

based on principle of maximum plastic work

$$\int_V\dot{A}\,dv\leq 0 \quad \Rightarrow \quad \sigma_{(1)}\approx\sigma_{(2)}$$

Stress distribution converges to specific one by plastic deformation.

Lower bound theorem of Limit Analysis

- Statically admissible stress field

$\nabla\cdot\sigma+\mathbf{f}=\mathbf{0}$ in V, $\quad \mathbf{n}\cdot\sigma=\rho\mathbf{t}_o$ on Sσ

$f(\sigma)<0$ in V

- Theorem

If statically admissible stress field can be found for external load $\rho\mathbf{t}_o$, the body is safe for it.

$\rho\leq\rho_{exact}$

(1) Kinematic condition is not considered.

(2) Maximization of lower solution is necessary.

Proof

- Suppose exact solutions of σ_* , $\dot{\varepsilon}_*^p$ & ρ_* .

$$\int_v \sigma_* : \dot{\varepsilon}_*^p \, dv = \int_v \mathbf{f} \cdot \dot{\mathbf{u}}_* \, dv + \int_{S\sigma} \rho_* \mathbf{t}_o \cdot \dot{\mathbf{u}}_* \, ds$$

weak form of statically admissible stress

$$\int_v \sigma : \dot{\varepsilon}_*^p \, dv = \int_v \mathbf{f} \cdot \dot{\mathbf{u}}_* \, dv + \int_{S\sigma} \rho \mathbf{t}_o \cdot \dot{\mathbf{u}}_* \, ds$$

From two equations

$$\int_v (\sigma_* - \sigma) : \dot{\varepsilon}_*^p \, dv = (\rho_* - \rho) \int_{S\sigma} \mathbf{t}_o \cdot \dot{\mathbf{u}}_* \, ds \geq 0$$

⇨ $\rho_* - \rho \geq 0$

Upper bound theorem of Limit Analysis

- Kinematically admissible velocity field

$$\dot{\varepsilon} = \frac{1}{2}\left\{\nabla \dot{\mathbf{u}} + (\nabla \dot{\mathbf{u}})^T\right\} \quad \text{in V,} \qquad \dot{\mathbf{u}} = \dot{\mathbf{u}}_o \quad \text{on Su}$$

- Theorem

Load factor calculated by setting rate of external load work equal to rate of internal dissipation energy

$$\int_v D(\dot{\varepsilon}^p) dv = \int_v \mathbf{f} \cdot \dot{\mathbf{u}} \, dv + \int_{S\sigma} \rho \mathbf{t}_o \cdot \dot{\mathbf{u}} \, ds$$

is greater than the exact one, $\rho_{exact} \leq \rho$.

(1) Equilibrium equation is not considered.

(2) Minimization of upper bound solution is necessary.

Internal dissipation energy

- Definition

$$D(\dot{\varepsilon}^p) = \sigma : \dot{\varepsilon}^p$$

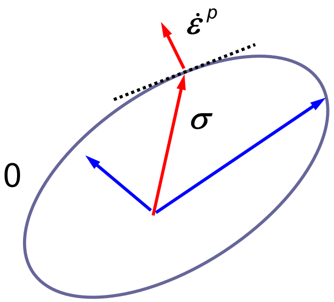
Stress σ is uniquely determined for $\dot{\varepsilon}^p$ by principle of maximum plastic work.

- Property

(1) convex function of $\dot{\varepsilon}^p$

$$D(\dot{\varepsilon}_{(2)}^p) - D(\dot{\varepsilon}_{(1)}^p) - \frac{\partial D}{\partial \dot{\varepsilon}_{(1)}^p} \cdot (\dot{\varepsilon}_{(2)}^p - \dot{\varepsilon}_{(1)}^p) \geq 0$$

(2) $D(2\dot{\varepsilon}^p) = D(\dot{\varepsilon}^p)$



Proof of upper bound theorem

- Suppose exact solutions of σ_* & ρ_* .

weak form of statically admissible stress

$$\int_v \sigma_* : \dot{\varepsilon}^p \, dv = \int_v \mathbf{f} \cdot \dot{\mathbf{u}} \, dv + \int_{S\sigma} \rho_* \mathbf{t}_o \cdot \dot{\mathbf{u}} \, ds$$

upper bound calculation

$$\int_v D(\dot{\varepsilon}^p) dv = \int_v \mathbf{f} \cdot \dot{\mathbf{u}} \, dv + \int_{S\sigma} \rho \mathbf{t}_o \cdot \dot{\mathbf{u}} \, ds$$

From two equations,

$$\int_v (\sigma_* - \sigma) : \dot{\varepsilon}^p \, dv = (\rho_* - \rho) \int_{S\sigma} \mathbf{t}_o \cdot \dot{\mathbf{u}}_* \, ds \leq 0$$

⇨ $\rho_* - \rho \leq 0$

Application of upper bound theorem

- Upper bound theorem

$$\min\left\{\rho \,\middle|\, \begin{array}{l} \rho\int_{S\sigma}\mathbf{t}_o\cdot\dot{\mathbf{u}}\,ds+\int_v\mathbf{f}\cdot\dot{\mathbf{u}}\,dv=\int_v \mathrm{D}\left(\dot{\boldsymbol{\varepsilon}}^p\right)dv \\ \text{sub. to}\quad \dot{\boldsymbol{\varepsilon}}^p=\frac{1}{2}\left(\nabla\dot{\mathbf{u}}+\nabla\dot{\mathbf{u}}^T\right)\text{ in v,} \\ \qquad\qquad \dot{\mathbf{u}}=\dot{\mathbf{u}}_o\text{ on s}_\mathrm{u} \end{array}\right\}$$

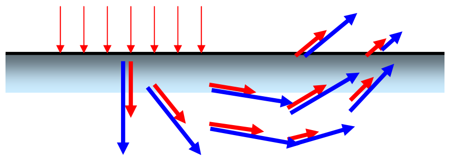
- Governing equation

$$\rho=\frac{\int_v \mathrm{D}\left(\dot{\boldsymbol{\varepsilon}}^p\right)dv-\int_v\mathbf{f}\cdot\dot{\mathbf{u}}\,dv}{\int_{S\sigma}\mathbf{t}_o\cdot\dot{\mathbf{u}}\,ds}$$

Property of object function

- Homogeneous function

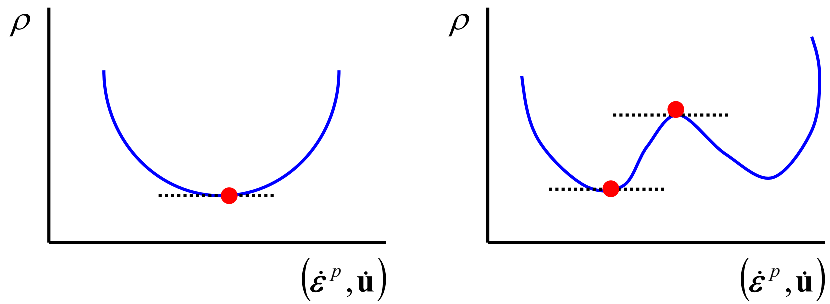
$$\int_v \mathrm{D}\left(2\dot{\boldsymbol{\varepsilon}}^p\right)dv-\int_v\mathbf{f}\cdot\left(2\dot{\mathbf{u}}\right)dv=\int_v 2\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^p\,dv-\int_v 2\mathbf{f}\cdot\dot{\mathbf{u}}\,dv$$
$$=2\left(\int_v\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^p\,dv-\int_v\mathbf{f}\cdot\dot{\mathbf{u}}\,dv\right)=2\left(\int_v \mathrm{D}\left(\dot{\boldsymbol{\varepsilon}}^p\right)dv-\int_v\mathbf{f}\cdot\dot{\mathbf{u}}\,dv\right)$$



Relative magnitude & direction of $\dot{\boldsymbol{\varepsilon}}^p$ are important in assessing limit load

Property of object function

- Convexity property on function



Global minimization = Local minimization

Search of “ $\delta\rho=0$ “

Minimization of upper bound solution

- Upper bound theorem

$$\min\left\{\rho \,\middle|\, \begin{array}{l} \rho=\int_v \mathrm{D}\left(\dot{\boldsymbol{\varepsilon}}^p\right)dv-\int_v\mathbf{f}\cdot\dot{\mathbf{u}}\,dv \\ \text{sub. to}\quad \int_{S\sigma}\mathbf{t}_o\cdot\dot{\mathbf{u}}\,ds=1 \\ \qquad\qquad \dot{\boldsymbol{\varepsilon}}^p=\frac{1}{2}\left(\nabla\dot{\mathbf{u}}+\nabla\dot{\mathbf{u}}^T\right)\text{in v,} \\ \qquad\qquad \dot{\mathbf{u}}=\dot{\mathbf{u}}_o\text{ on s}_\mathrm{u} \end{array}\right\}$$

- Functional method for minimization with condition

$$\rho=\int_v \mathrm{D}\left(\dot{\boldsymbol{\varepsilon}}^p\right)dv-\int_v\mathbf{f}\cdot\dot{\mathbf{u}}\,dv+\mu\left(\int_{S\sigma}\mathbf{t}_o\cdot\dot{\mathbf{u}}\,ds-1\right)$$

$\Rightarrow\quad \delta\rho=0\quad\text{for }\left(\delta\dot{\mathbf{u}},\delta\mu\right)$ $\qquad$ μ :Lagrange multiplier

Simultaneous equation in minimization

- Upper bound theorem

$$\rho = \int_v D(\dot{\boldsymbol{\varepsilon}}^p)\,dv - \int_v \mathbf{f}\cdot\dot{\mathbf{u}}\,dv + \mu\left(\int_{S\sigma} \mathbf{t}_o\cdot\dot{\mathbf{u}}\,ds - 1\right)$$

Limit load factor

$$\Rightarrow \begin{cases} \int_v \boldsymbol{\sigma}:\delta\dot{\boldsymbol{\varepsilon}}^p\,dv - \int_v \mathbf{f}\cdot\delta\dot{\mathbf{u}}\,dv + \mu\int_{S\sigma} \mathbf{t}_o\cdot\delta\dot{\mathbf{u}}\,ds = 0 \\ \int_{S\sigma} \mathbf{t}_o\cdot\dot{\mathbf{u}}\,ds - 1 = 0 \end{cases}$$

It is same with equilibrium equation

Minimization process derives the equation of equilibrium which is not considered in upper bound theorem.

All necessary equations are taken into account in boundary value problem and the exact solution can be obtained.

Constraint condition for constitutive eqn.

- von Mises criteria

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{\dot{e}}{\sqrt{2}\sigma_o}\mathbf{s} \quad \Rightarrow \quad \dot{\varepsilon}_v{}^p = \frac{\dot{e}}{\sqrt{2}\sigma_o}tr\mathbf{s} = 0$$

- Functional to minimize an object function with conditions

$$\rho = \int_v D(\dot{\boldsymbol{\varepsilon}}^p)\,dv - \int_v \mathbf{f}\cdot\dot{\mathbf{u}}\,dv + \mu\left(\int_{S\sigma} \mathbf{t}_o\cdot\dot{\mathbf{u}}\,ds - 1\right) + \int_v \kappa\dot{\varepsilon}_v{}^p\,dv = 0$$

$$\Rightarrow \begin{cases} \int_v (\mathbf{s}+\kappa\mathbf{I}):\delta\dot{\boldsymbol{\varepsilon}}^p\,dv - \int_v \mathbf{f}\cdot\delta\dot{\mathbf{u}}\,dv + \mu\int_{S\sigma} \mathbf{t}_o\cdot\delta\dot{\mathbf{u}}\,ds = 0 \\ \int_{S\sigma} \mathbf{t}_o\cdot\dot{\mathbf{u}}\,ds - 1 = 0 \\ \int_v \delta\kappa\,\dot{\varepsilon}_v{}^p\,dv \end{cases}$$

$$\boldsymbol{\sigma} = \mathbf{s} + p\mathbf{I} = \frac{\sqrt{2}\sigma_o}{\dot{e}}\dot{\boldsymbol{\varepsilon}}^p + p\mathbf{I}$$

Obtained equation matches the rigid plastic constitutive equation

Rigid plastic FEM

- Finite element discretization

$\dot{\mathbf{u}} = N\dot{\mathbf{U}}$ $\quad$ $\dot{\boldsymbol{\varepsilon}}^p = B\dot{\mathbf{U}}$ $\quad$ $\dot{\varepsilon}_v{}^p = \mathbf{m}^T B\dot{\mathbf{U}}$

N :shape function $\quad$ B :kinematic matrix $\quad$ $\mathbf{m}$:transfer vector

$$\int_{S\sigma} \mathbf{t}_o\cdot\dot{\mathbf{u}}\,ds - 1 = 0 \quad \Rightarrow \quad \int_{S\sigma} \mathbf{t}_o{}^T N_1\,ds\,\dot{\mathbf{U}} - 1 = \mathbf{F}^T\dot{\mathbf{U}} - 1 = 0$$

$$\int_v \delta\kappa\,\dot{\varepsilon}_v{}^p\,dv = 0 \quad \Rightarrow \quad \delta\boldsymbol{\kappa}^T\int_v \mathrm{N}_2{}^{\mathrm{T}}\mathrm{m}^{\mathrm{T}}\mathrm{B}\,dv\,\dot{\mathbf{U}} = \delta\boldsymbol{\kappa}^T L\dot{\mathbf{U}} = 0$$

$$\int_v (\mathbf{s}+\kappa\mathbf{I}):\delta\dot{\boldsymbol{\varepsilon}}^p\,dv - \int_v \mathbf{f}\cdot\delta\dot{\mathbf{u}}\,dv + \mu\int_{S\sigma} \mathbf{t}_o\cdot\delta\dot{\mathbf{u}}\,ds = 0$$

$$\Rightarrow \delta\dot{\mathbf{U}}^T\left[\int_v \sqrt{2}\sigma_o\frac{B^TQB}{\dot{e}}\,dv\,\dot{\mathbf{U}} + \int_v B^T\mathbf{m}N_2\,dv\,\boldsymbol{\kappa} - \int_v N_3{}^T\mathbf{f}\,dv + \mu\int_{S\sigma} N_1{}^T\mathbf{t}_o ds\,\mu\right] = 0$$

Rigid plastic FEM

- Simultaneous equations

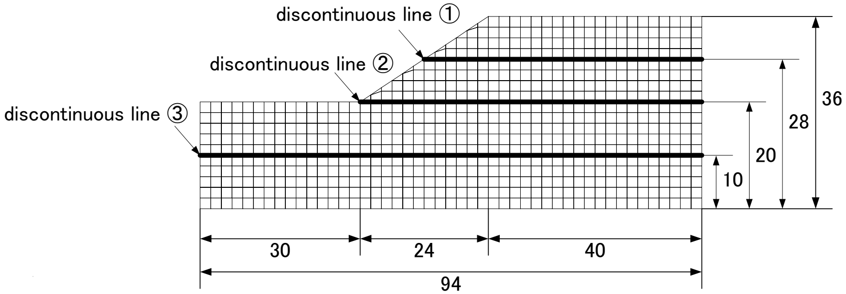
$$\begin{cases} \left(\int_v \sqrt{2}\sigma_o\frac{B^TQB}{\dot{e}}\,dv\right)\dot{\mathbf{U}} + \left(\int_v B^T\mathbf{m}N_2\,dv\right)\boldsymbol{\kappa} + \left(\int_{S\sigma} N_1{}^T\mathbf{t}_o ds\right)\mu = \int_v N_3{}^T\mathbf{f}\,dv \\ \int_v \mathrm{N}_2{}^{\mathrm{T}}\mathbf{m}^T\mathrm{B}\,dv\,\dot{\mathbf{U}} = \mathbf{0} \\ \int_{S\sigma} \mathbf{t}_o{}^T N_1\,ds\,\dot{\mathbf{U}} - 1 = 0 \end{cases}$$

velocity $\quad$ mean stress $\quad$ load factor

$$\Rightarrow \begin{cases} \mathbf{K}(\dot{e})\dot{\mathbf{U}} + \mathbf{L}^T\boldsymbol{\kappa} + \mathbf{F}\mu = \mathbf{0} \\ \mathbf{L}\dot{\mathbf{U}} = \mathbf{0} \\ \mathbf{F}^T\dot{\mathbf{U}} = 1 \end{cases}$$

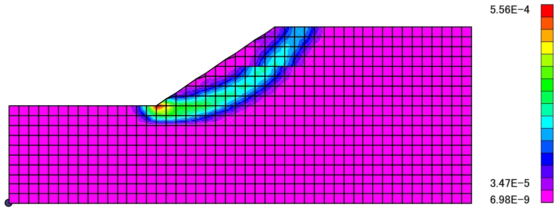
Symmetric tensor

Slope stability analysis



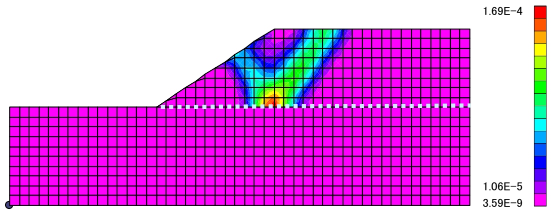
接触ジョイントのせん断抵抗角 ϕ	20°
粘着力 c	20kPa
単位体積重量 γ_t	$18kN/m^3$
不連続線のせん断抵抗角 ϕ_d	0°
不連続線の粘着力 c_d	10kPa

With and without analysis of seam layer



Fs=1.307

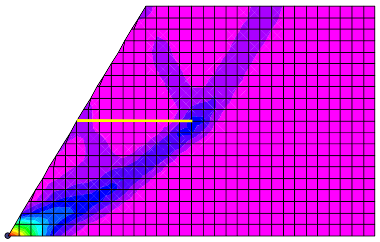
Simple slope



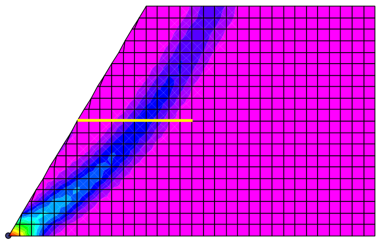
Fs=0.9376

Seam weak layer is included

Effect of nailing strength



High strength of nailing



Low strength of nailing

接触ジョイントのせん断抵抗角 ϕ_c	35°
接触ジョイントの粘着力 c_c	20kPa
節点ジョイントの粘着力 c_p	0.01kPa
対策工の降伏軸力 Ny	200kN
対策工の降伏曲げモーメント My	0.01kN･m
対策工の直径 D	0.2m